

Irrational constants in positronium decays

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We establish irrational constants, that contribute to the positronium lifetime at $O(\alpha)$ and $O(\alpha^2)$ order. In particular we show, that a new type of constants appear, which are not related to Euler–Zagier sums or multiple ζ values.

1. Introduction

Most of the multi-loop analytical calculations in quantum field theories have been done for so-called single-scale problems. This means that the evaluated integrals are basically expressed as numerical constants up to a trivial scale factor. Examples of such problems include almost all renormalization group calculations, evaluations of the critical exponents, anomalous magnetic moments of the electron and the muon, matching calculations in effective theories (e.g. HQFT, NRQFT) and many others.

Usually analytical results involve the so-called Euler–Zagier (EZ) sums of the form

$$\sum_{n_1 > n_2 > \dots > n_k} \frac{(\pm 1)^{n_1}}{n_1^{a_1}} \dots \frac{(\pm k)^{n_k}}{n_1^{a_k}} \quad (1)$$

or more generally multiple polylogarithms

$$\sum_{n_1 > n_2 > \dots > n_k} \frac{z_1^{n_1}}{n_1^{a_1}} \dots \frac{z_k^{n_k}}{n_1^{a_k}} \quad (2)$$

where z_1, \dots, z_k are some parameters and a_1, \dots, a_k are positive integers. The sum $a_1 + a_2 + \dots + a_k$ is called the *weight* in such a case.

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The above definitions include e.g. well-known irrationalities like ζ functions $\zeta(a)$, $\zeta(a, b)$, \dots , (poly)logarithms $\text{Li}_a(1/2)$, $\ln 2$, \dots and “sixth root of unity” constants $\text{Ls}_j^{(k)}(\pi/3)$, $\text{Ls}_j^{(k)}(2\pi/3)$, \dots . There is no doubt, that by consideration of more complicated problems and in higher loops some new constants will appear. As examples we can mention some elliptic integrals (see e.g. [1,2,3]).

In this paper we concentrate on a very important single-scale problem: the total width of positronium decay in QED. Positronium (Ps), the lightest known atom, provides an ultra-pure laboratory for high-precision tests of QED. In fact, thanks to the smallness of the electron mass m relative to typical hadronic mass scale, its theoretical description is not plagued by strong interaction uncertainties and its properties, such as decay widths and energy levels can be calculated perturbatively in non-relativistic QED (NRQED) [4] with very high precision.

Ps comes in two ground states, 1S_0 parapositronium (p -Ps) and 3S_1 orthopositronium (o -Ps), which decay to two and three photons, respectively.

2. Orthopositronium

In this section we are concerned with the lifetime of o -Ps, which has been the subject of a vast number of theoretical and experimental investigations. Its first precision measurement [5], of 1968, had to wait nine years to be compared with first complete one-loop calculation [6], which came two decades after the analogous calculation for p -Ps [7] being considerably simpler owing to the two-body final state. In the year 1987, the Ann Arbor group [8] published a measurement that exceeded the theoretical prediction available by ten experimental standard deviations. This is so-called o -Ps lifetime puzzle triggered an avalanche of both experimental and theoretical activities, which eventually resulted in what now appears to be the resolution of this puzzle. In fact, the 2003 measurements at Ann Arbor [9] and Tokio [10]

$$\begin{aligned}\Gamma(\text{Ann Arbor}) &= 7.0404(10)(8) \mu s^{-1}, \\ \Gamma(\text{Tokyo}) &= 7.0396(12)(11) \mu s^{-1},\end{aligned}\quad (3)$$

agree mutually and with the present theoretical prediction,

$$\Gamma(\text{theory}) = 7.039979(11) \mu s^{-1}. \quad (4)$$

The latter is evaluated from

$$\begin{aligned}\Gamma(\text{theory}) &= \Gamma_0 \left[1 + A \frac{\alpha}{\pi} + \frac{\alpha^2}{3} \ln \alpha \right. \\ &\quad + B \left(\frac{\alpha}{\pi} \right)^2 - \frac{3\alpha^3}{2\pi} \ln^2 \alpha \\ &\quad \left. + C \frac{\alpha^3}{\pi} \ln \alpha \right],\end{aligned}\quad (5)$$

where [11]

$$\Gamma_0 = \frac{2}{9}(\pi^2 - 9) \frac{m\alpha^6}{\pi} \quad (6)$$

is the LO result. The leading logarithmically enhanced $\mathcal{O}(\alpha^2 \ln \alpha)$ and $\mathcal{O}(\alpha^3 \ln^2 \alpha)$ terms were found in Refs. [12,13] and Ref. [14], respectively. The coefficients $A = -10.286606(10)$ [6,12,15,16,17], $B = 45.06(26)$ [16], and $C = -5.51702455(23)$ [18] are only available in numerical form so far. Comprehensive reviews of the present experimental and theoretical status of o -Ps may be found in Ref. [19].

Given the fundamental importance of Ps for atomic and particle physics, it is desirable to complete our knowledge of the QED prediction in Eq. (5). Since the theoretical uncertainty is presently dominated by the errors in the numerical evaluations of the coefficients A , B , and C , it is an urgent task to find them in analytical form, in terms of irrational numbers, which can be evaluated with arbitrary precision. In this Letter, this is achieved for A and C . The case of B is beyond the scope of presently available technology, since it involves two-loop five-point functions to be integrated over a three-body phase space. The quest for an analytic expression for A is a topic of old vintage: about 25 years ago, some of the simpler contributions to A , due to self-energy and outer and inner vertex corrections, were obtained analytically [21], but further progress then soon came to a grinding halt.

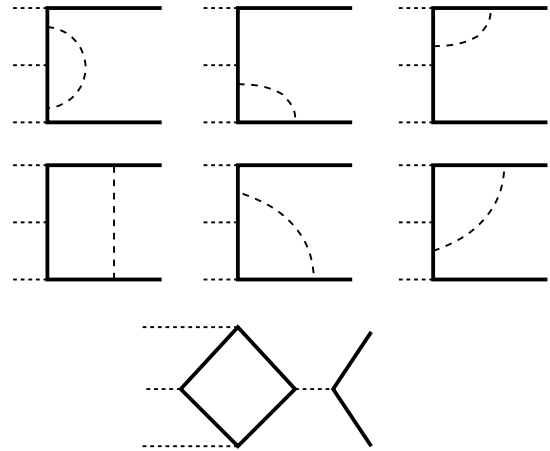


Figure 1. Feynman diagrams contributing to the total decay width of o -Ps at $\mathcal{O}(\alpha)$. Self-energy diagrams are not shown. Dashed and solid lines represent photons and electrons, respectively.

The $\mathcal{O}(\alpha)$ contribution in Eq. (5), $\Gamma_1 = \Gamma_0 A \alpha / \pi$, is due to the Feynman diagrams where a virtual photon is attached in all possible ways

to the tree-level diagrams, with three real photons linked to an open electron line with threshold kinematics. Such diagrams are shown in Fig. 1.

After angular integration over three-photon phase space

$$\int [dk_1][dk_2][dk_3]\delta(k_1 + k_2 + k_3 - q) \quad (7)$$

we can rewrite the one-loop contribution to the width as (see [17])

$$\begin{aligned} \Gamma_1 = & \frac{m\alpha^7}{36\pi^2} \int_0^1 \frac{dx_1}{x_1} \frac{dx_2}{x_2} \frac{dx_3}{x_3} \delta(2 - x_1 - x_2 - x_3) \\ & \times [F(x_1, x_3) + \text{perm.}], \end{aligned} \quad (8)$$

where x_i , with $0 \leq x_i \leq 1$, is the energy of photon i in the o -Ps rest frame normalized by its maximum value, the delta function ensures energy conservation, and *perm.* stands for the other five permutations of x_1, x_2, x_3 .

The function F includes dilogarithm and arctangent functions as given in [17]. For illustration, we just mention, that the above expression, after re-parametrization, consists of integrals of the following type

$$\begin{aligned} & \frac{P(x_1, x_2, x_3)}{Q(x_1, x_2, x_3)} \int_0^1 \frac{dy \ln(x_1 + (1 - x_1)y^2)}{(1 - x_1)x_3 - x_1(1 - x_3)y^2}, \\ & \frac{P(x_1, x_2, x_3)}{Q(x_1, x_2, x_3)} \int_0^1 \frac{dy \ln(x_1 + (1 - x_1)y^2)}{x_1x_3 - (1 - x_1)(1 - x_3)y^2}, \end{aligned}$$

with $P, Q, P'Q'$ being some polynomials.

The analytical integration of the above expressions is rather tedious and requires a number of tricks, e.g. expansion in series. Only a few integrals could be done straightforwardly, e.g. with *Mathematica* or *Maple*. However, we established all irrational constants in terms of which the complete one-loop correction can be expressed. These include among others usual EZ sums up to weight four, including e.g.

$$\ln 2, \quad \zeta(n), \quad \text{Li}_4\left(\frac{1}{2}\right), \quad \text{etc.}$$

and some additional constants of new type. At weight one, we have

$$\ln(R), \quad \text{where} \quad R = \frac{\sqrt{2} - 1}{\sqrt{2} + 1}$$

and up to weight four our basis includes the following constants

$$\begin{aligned} & \text{Li}_2\left(\frac{1}{3}\right), \quad \text{Li}_4\left(\frac{1}{3}\right), \quad \text{Li}_4\left(-\frac{1}{3}\right), \\ & \text{Li}_3\left(\frac{1}{\sqrt{2}}\right), \quad \text{Li}_3(R), \quad \text{S}_{1,2}(R), \\ & \text{Li}_4(\pm R), \quad \text{S}_{1,3}(\pm R), \quad \text{S}_{2,2}(\pm R), \end{aligned}$$

with $\text{S}_{a,b}$ being the generalized polylogarithm

$$\text{S}_{a,b}(x) = \frac{(-1)^{a+b-1}}{(a-1)!b!} \int_0^1 \frac{dt}{t} \ln^{a-1} t \ln^b(1 - tx).$$

Unfortunately, not all integrals can be computed analytically. In more complicated cases, the integrations are not separated after expansion into infinite series. We then rely on the PSLQ algorithm [26], which allows one to reconstruct the representation of a numerical result known to very high precision in terms of a linear combinations of a set of constants with rational coefficients, if that set is known beforehand. The experience gained with the explicit solution of the simpler integrals helps us to exhaust the relevant set. In order for PSLQ to work in our applications, the numerical values of the integrals must be known up to typically 150 decimal figures.

3. Parapositronium

Let us now turn to the case of parapositronium. Its total width was recently measured to be [28]

$$\Gamma_p(\text{exp}) = 7990.9 \mu\text{s}^{-1}. \quad (9)$$

At present, the following radiative corrections within NRQED are available:

$$\begin{aligned} \Gamma_p = & \frac{\alpha^5 m_e}{2} \left\{ 1 + \frac{\alpha}{\pi} \left(\frac{\pi^2 - 20}{4} \right) \right. \\ & + \frac{\alpha^2}{\pi^2} (-2\pi^2 \ln \alpha + A_p) + \frac{\alpha^3}{\pi} \left(-\frac{3}{2} \ln^2 \alpha \right. \\ & \left. \left. + \left(\frac{533}{90} - \frac{\pi^2}{2} + 10 \ln 2 \right) \ln \alpha \right) \right\}. \end{aligned}$$

The first-order corrections were obtained in [7], while the logarithmically enhanced terms were computed in [12,13]. Here the constant $A_p = 5.12443(33)$ is known only numerically [20] and our next goal is to establish the irrational constants that contribute to this quantity.

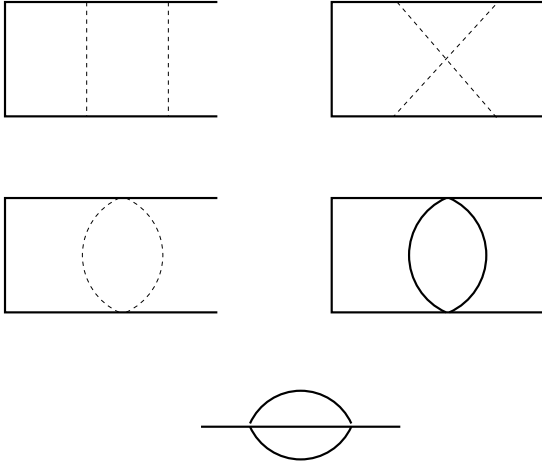


Figure 2. Diagrams contributing to the decay width of p -Ps at $O(\alpha^2)$ and their reduction to simpler master integrals. Dashed and solid lines represent massless and massive lines, respectively.

This quantity receives contributions from two-loop diagrams of e^+e^- annihilation into two photons in threshold kinematics. However, the generic planar and non-planar diagrams (see Fig. 2, upper row) can be reduced via integration by parts to simpler integrals (Fig. 2, middle row). These, in turn, as we shall see, contain constants that are related to the sunset diagram (Fig. 2, bottom row) at very special kinematics, namely when the external momentum q is restricted by $q^2 = -m^2$. The sunset diagrams with such kinematics have been considered in great detail in [1]. In particular the result for the sunset is expressed

in terms of special sums of elliptic nature,

$$\sum_{n=1}^{\infty} (-1)^n \frac{\binom{2n}{n}}{\binom{4n}{2n}} \left\{ \phi, \frac{\phi}{n}, \frac{1}{n^2} \right\}, \quad (10)$$

which we can call a_ϕ , $a_{\phi 1}$ and a_2 , respectively, and other sums

$$\sum_{n=1}^{\infty} \frac{(-16)^n}{\binom{2n}{n} \binom{4n}{2n}} \left\{ 1, \frac{1}{n} \right\}, \quad (11)$$

which we call b_0 and b_1 . In (10), ϕ stands for

$$\phi = S_1(n-1) - 3S_1(2n-1) + 2S_1(4n-1),$$

with $S_a(n) = \sum_{j=1}^n 1/j^a$ being a harmonic sum.

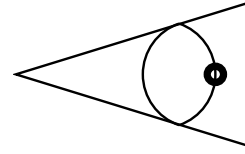


Figure 3. Example of vertex diagram J , contributing to the decay width of p -Ps. All lines have mass m . The dot on a line means the square of the propagator.

Starting from (10) and (11), one can construct sums of higher weights, e.g. a_3 , $a_{\phi 2}$, b_3 , etc. With such constructed sums, we evaluate more complicated diagrams, including vertexes and boxes. We illustrate it evaluating diagram J shown in Fig. 3. The result is

$$J = \frac{9}{16}\zeta(3) - \frac{1}{8}a_3 - \frac{1}{8}a_{\phi 2} - \frac{1}{32}b_3 \quad (12)$$

and a similar result follows for the box diagrams of Fig. 2. Formula (12) shows the deep relation of the vertex diagram with the sunset diagram (in fact such relation follows from the differential equations).

Concluding this section we want to mention that there are relations between the above sums and also their relation to the elliptic integrals has been found in [1].

4. Conclusions

Thus, we established the analytical structure of the results for the next unknown corrections both for ortho- and parapositronium lifetimes. We found that new constants, that are not related to the Euler–Zagier sums appear in both cases,

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